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ON ALMOST STRONGLY θ-b-CONTINUOUS FUNCTIONS

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Abstract. We introduce a new class of functions called almost strongly θ-b-continuous function which is a generalization of strongly θ-continuous functions and strongly θ-b-continuous functions. Some characterizations and several properties concerning almost strongly θ-b-continuous function are obtained.

1. introduction

A subset A of a topological space X is b-open [2] or sp-open [7] if $A \subseteq Int(Cl(A)) \cup Cl(Int(A))$. A function $f : X \to Y$ is called b-continuous [8] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a b-open U containing x such that $f(U) \subseteq V$, which is equivalent to say that the preimage $f^{-1}(V)$ of each open set V of Y is b-open in X. Recently, Park [16] introduced and investigated the notion of strongly θ-b-continuous functions which is stronger than b-continuous, moreover see [3, 4, 5].The purpose of the present paper is to introduce and investigate a weaker form of strongly θ-b-continuity called almost strongly θ-b-continuous function.

For the benefit of the reader we recall some basic definitions and known results. Throughout the present paper, the space X and Y (or (X, τ) and (Y, σ)) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let A be a subset of X. The closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively.

The complement of an b-open set is called b-closed. The smallest b-closed set containing $A \subseteq X$ is called the b-closure, of A and shall be denoted by $bCl(A)$. The union of all b-open set of X contained in A is called the b-interior of A and is denoted by $bInt(A)$. A subset A is said to be b-regular if it is b-open and b-closed. The family of all b-open (resp; b-closed, b-regular, open) subsets of a space X is denoted by $BO(X)$ (resp; $BC(X)$, $BR(X)$,$O(X)$ respectively) and the collection of all b-open subsets of X containing a fixed point x is denoted by $BO(X, x)$. The sets $O(X, x)$ and $BR(X, x)$ are defined analogously.

A point $x \in X$ is called a θ-cluster point of A if $Cl(U) \cap A \neq \phi$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure [18] of A and is denoted by $Cl_\theta(A)$. A subset A is said to be θ-closed [18] if $Cl_\theta(A) = A$. The complement of a θ-closed set is said to be θ-open.

A point x of X is called a b-θ-cluster [16] point of A if $bCl(U) \cap A \neq \phi$ for every $U \in BO(X, x)$. The set of all b-θ-cluster points of A is called b-θ-closure of A and is denoted by $bCl_\theta(A)$. A subset A is said to be

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b-θ-closed if $A = bCl_\theta(A)$. The complement of a b-θ-closed set is said to be b-θ-open set.

A subset A of X is called regular open (regular closed) if $A = Int(Cl(A))$ ($A = Cl(Int(A))$). The δ-interior of a subset A of X is the union of all regular open sets of X contained in A and it is denoted by δ-$Int(A)$ [18]. A subset A is called δ-open if $A = \delta$-$Int(A)$. The complement of a δ-open set is called δ-closed. The δ-closure of a set A in a space (X,τ) is defined by $\{x \in X : A \cap Int(Cl(B)) \neq \phi,\ B \in \tau$ and $x \in B\}$ and it is denoted by δ-$Cl(A)$.

2. CHARACTERIZATIONS

Definition 2.1. A function $f : X \to Y$ is said to be almost strongly θ-b-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in BO(X,x)$ such that $f(bCl(U)) \subseteq Int(Cl(V))$.

Definition 2.2. [16] A function $f : X \to Y$ is said to be strongly θ-b-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in BO(X,x)$ such that $f(bCl(U)) \subseteq V$.

Then it is clear that every strongly θ-b-continuous is almost strongly θ-b-continuous but the converse is not true.

Definition 2.3. [14] A function $f : X \to Y$ is said to be strongly θ-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(Cl(U)) \subseteq V$.

Example 2.4. *Let* $X = \{a,b,c\}$, $(X,\tau) = \{X,\phi,\{a\},\{a,b\}\}$ *with* $BO(X,\tau) = \{X,\phi,\{a\},\{a,b\},\{a,c\}\}$ *and* $(X,\sigma) = \{X,\phi,\{a\}\}$. *And* $f : (X,\tau) \to (X,\sigma)$ *be defined by* $f(a) = b$, $f(b) = c$ *and* $f(c) = a$. *Then* f *is almost strongly* θ-b-*continuous but it is not strongly* θ-b-*continuous . Since the open set* $V = \{a\}$ *in* (X,σ) *containing* $f(c)$ *and there is no b-open set* U *in* (X,τ) *containing c such that* $f(bCl(U)) \subseteq V$.

Theorem 2.5. *For a function* $f : X \to Y$, *the following are equivalent:*

(1) f *is almost strongly* θ-b-*continuous;*
(2) $f^{-1}(V)$ *is* b-θ-*open in* X *for each regular open set* V *of* Y*;*
(3) $f^{-1}(F)$ *is* b-θ-*closed in* X *for each regular closed set* F *of* Y*;*
(4) *for each* $x \in X$ *and each regular open set* V *of* Y *containing* $f(x)$, *there exists* $U \in BO(X,x)$ *such that* $f(bCl(U)) \subseteq V$*;*
(5) $f^{-1}(V)$ *is* b-θ-*open in* X *for each* δ-*open set* V *of* Y*;*
(6) $f^{-1}(F)$ *is* b-θ-*closed in* X *for each* δ-*closed set* F *of* Y*;*
(7) $f(bCl_\theta(A)) \subseteq Cl_\delta(f(A))$ *for each subset* A *of* X*;*
(8) $bCl_\theta(f^{-1}(B)) \subseteq f^{-1}(Cl_\delta(B))$ *for each subset* B *of* Y.

Proof. (1) → (2): Let V be any regular open set of Y and $x \in f^{-1}(V)$. Then $V = int(clV)$ and $f(x) \in V$. Since f is almost strongly θ-b-continuous, there exists $U \in BO(X,x)$ such that $f(bCl(U)) \subseteq V$. Therefore, we have $x \in U \subseteq bCl(U) \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is b-θ-open in X.
(2) → (3): Let F be any regular closed set of Y. By (2), $f^{-1}(F) = X - f^{-1}(Y - F)$ is b-θ-closed in X.
(3) → (4): Let $x \in X$ and V be any regular open set of Y containing $f(x)$. By (3), $f^{-1}(Y - V) = X - f^{-1}(V)$ is b-θ-closed in X and so $f^{-1}(V)$ is a b-θ-open set containing x, there exists $U \in BO(X,x)$ such that $bCl(U) \subseteq f^{-1}(V)$. Therefore, we have $f(bCl(U)) \subseteq V$.
(4) → (5): Let V be any δ-open set of Y

and $x \in f^{-1}(V)$. There exists a regular open set G of Y such that $f(x) \in G \subseteq V$. By (4), there exists $U \in BO(X, x)$ such that $f(bCl(U)) \subseteq G$. Therefore, we obtain $x \in U \subseteq bCl(U) \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is b-θ-open in X.

(5) → (6): Let F be any δ-closed set of Y. Then $Y - F$ is b-θ-open in Y and by (5), $f^{-1}(F) = X - f^{-1}(Y - F)$ is b-θ-closed in X.

(6) → (7): Let A be any subset of X. Since $Cl_\delta(f(A))$ is δ-closed in Y, by (6) $f^{-1}(Cl_\delta(f(A)))$ is b-θ-closed in X. Let $x \notin f^{-1}(Cl_\delta(f(A)))$. There exists $U \in BO(X, x)$ such that $bCl(U) \cap f^{-1}(Cl_\delta(f(A))) = \phi$ and thus $bCl(U) \cap A = \phi$. Hence $x \notin bCl_\theta(A)$. Therefore, we have $f(bCl_\theta(A)) \subseteq Cl_\delta(f(A))$.

(7) → (8): Let B be any subset of Y. By (7), we have $f(bCl_\theta(f^{-1}(B))) \subseteq Cl_\delta(B)$ and hence $bCl_\theta(f^{-1}(B)) \subseteq f^{-1}(Cl_\delta(B))$.

(8) → (1): let $x \in X$ and V be any open set of Y containing $f(x)$. Then $G = Y - Int(Cl(V))$ is regular closed and hence δ-closed in Y. By (8), $bCl_\theta(f^{-1}(G)) \subseteq f^{-1}(Cl_\delta(G)) = f^{-1}(G)$ and hence $f^{-1}(G)$ is b-θ-closed in X. Therefore, $f^{-1}(Int(Cl(V)))$ is b-θ-open set containing x. There exists $U \in BO(X, x)$ such that $bCl(U) \subseteq f^{-1}(Int(Cl(V)))$. Therefore we obtain $f(bCl(U)) \subseteq Int(Cl(V))$. This shows that f is almost strongly θ-b-continuous. □

Definition 2.6. A subset A of a space X is said to be:

(1) α-open [12] if $A \subseteq Int(Cl(Int(A)))$;
(2) semi-open [9] if $A \subseteq Cl(Int(A))$;
(3) preopen [11] if $A \subseteq Int(Cl(A))$;
(4) β-open [2] if $A \subseteq Cl(Int(Cl(A)))$.

Theorem 2.7. *For a function $f : X \to Y$, the following are equivalent:*

(1) *f is almost strongly θ-b-continuous;*
(2) *$bCl_\theta(f^{-1}(V) \subseteq f^{-1}(Cl(V))$ for each β-open set V of Y;*
(3) *$bCl_\theta(f^{-1}(V) \subseteq f^{-1}(Cl(V))$ for each b-open set V of Y;*
(4) *$bCl_\theta(f^{-1}(V) \subseteq f^{-1}(Cl(V))$ for each semi-open set V of Y.*

Proof. (1) → (2): Let V be any β-open set of Y. Then by Theorem 2.4 in [1] $Cl(V)$ is regular closed in Y. Since f is almost strongly θ-b-continuous, $f^{-1}(Cl(V))$ is b-θ-closed in X and hence $bCl_\theta(f^{-1}(V) \subseteq f^{-1}(Cl(V))$.

(2) → (3): This is obvious since every b-open set is β-open.

(3) → (4): This is obvious since every semi-open set is b-open.

(4) → (1): Let F be any regular closed set of Y. Then F is semi-open in Y and by(4) $bCl_\theta(f^{-1}(F) \subseteq f^{-1}(Cl(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is b-θ-closed in X. Therefore f is almost strongly θ-b-continuous. □

Recall that a space X is said to be almost regular [15](resp; semi-regular) if for any regular open (resp; open) set U of X and each point $x \in U$, there exist a regular open set V of X such that $x \in V \subseteq Cl(V) \subseteq U$ (resp; $x \in V \subseteq U$).

Theorem 2.8. *For any function $f : X \to Y$, the following properties hold:*

(1) *If f is b-continuous and Y is almost regular, then f is almost strongly θ-b-continuous;*
(2) *If f is almost strongly θ-b-continuous and Y is semi-regular, then f is strongly θ-b-continuous;*

Proof. (1) Let $x \in X$ and V be any regular open set of Y containing $f(x)$. Since Y is almost regular, there exists an open set W such that $f(x) \in W \subseteq Cl(W) \subseteq V$. Since f is b-continuous, there exists $U \in BO(X, x)$ such that $f(U) \subseteq W$. We shall

show that $f(bCl(U)) \subseteq Cl(W)$. Suppose that $y \notin Cl(W)$. There exists an open neighborhood G of y such that $G \cap W = \phi$. Since f is b-continuous, $f^{-1}(G) \in BO(X)$ and $f^{-1}(G) \cap U = \phi$ and hence $f^{-1}(G) \cap bCl(U) = \phi$. Therefore, we obtain $G \cap f(bCl(U)) = \phi$ and $y \notin f(bCl(U))$. Consequently. we have $f(bCl(U)) \subseteq Cl(W) \subseteq V$.
(2) Let $x \in X$ and V be any open set of Y containing $f(x)$. Since Y is semi-regular, there exists a regular open set W such that $f(x) \in W \subseteq V$. Since f is almost strongly θ-b-continuous, there exists $U \in BO(X, x)$such that $f(bCl(U) \subseteq W$. Therefore, we have $f(bCl(U) \subseteq V$. □

Definition 2.9. A topological space X is said to be b^*-regular (resp; b-regular [16], almost b-regular) if for each $F \in BC(X)$ (resp; $F \in C(X)$,F regular closed) and each $x \notin F$, there exist disjoint b-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 2.10. *For a topological space X, the following are equivalent:*

1. *X is b^*-regular (resp; b-regular [16]);*
2. *For each $U \in BO(X, x)$ (resp; $U \in O(X, x)$), there exists $V \in BO(X, x)$ such that $x \in V \subseteq bCl(V) \subseteq U$.*

It is Known that a function $f : X \to Y$ is almost continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq Int(Cl(V))$. Long and Herrington [10] proved that $f : X \to Y$ is almost continuous if and only if the inverse image of every regular open set in Y is open in X.

Theorem 2.11. *(1) If a continuous function $f : X \to Y$ is almost strongly θ-b-continuous then X is almost b-regular.*
(2) If $f : X \to Y$ is almost continuous and X is b-regular then f is almost strongly θ-b-continuous.

Proof. (1) Let $f : X \to Y$ be the identity. Then f is continuous and hence almost strongly θ-b-continuous. For any regular open set U of X and any points $x \in U$, we have $f(x) = x \in U$ and there exists $G \in BO(X, x)$ such that $f(bCl(G)) \subseteq U$. Therefore, we have $x \in G \subseteq bCl(G) \subseteq U$ and hence X is almost b-regular.
(2) Suppose that $f : X \to Y$ is almost continuous and X is b-regular. For each $x \in X$ and any regular open set V containing $f(x)$, $f^{-1}(V)$ is an open set of X containing x. Since X is b-regular there exists $U \in BO(X, x)$ such that $x \in U \subseteq bCl(U) \subseteq f^{-1}(V)$. Therefore, we have $f(bCl(U)) \subseteq V$. This shows that f is almost strongly θ-b-continuous. □

Theorem 2.12. [16] *Let A and B be any subset of a space X. Then the following properties hold:*

1. *$A \in BR(X)$ if and only if A is b-θ-open and b-θ-closed;*
2. *$x \in bCl_\theta(A)$ if and only if $V \cap A \neq \phi$ for each $V \in BR(X, x)$;*
3. *$A \in BO(X)$ if and only if $bCl(A) \in BR(X)$;*
4. *$A \in BC(X)$ if and only if $bInt(A) \in BR(X)$;*
5. *$A \in BO(X)$ if and only if $bCl(A) = bCl_\theta(A)$;*
6. *A is b-θ-open in X if and only if for each $x \in A$ there exists $V \in BR(X)$ such that $x \in V \subseteq A$.*

Lemma 2.13. *A subset U of a space X is b-θ-open in X if and only if for each $x \in U$, there exists b-open set W with $x \in W$ such that $bCl(W) \subseteq U$.*

Theorem 2.14. *For a function $f : X \to Y$, the following are equivalent:*

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(1) *f is almost strongly θ-b-continuous;*
(2) *for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists a b-θ-open set U containing x such that $f(U) \subseteq V$;*
(3) *for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists a b-open set W containing x such that $f(bCl_\theta(W)) \subseteq V$.*

Proof. (1) $\rightarrow$ (2): Let $x \in X$ and let V be any regular open subset of Y with $f(x) \in V$. Since f is almost strongly θ-b-continuous, $f^{-1}(V)$ is b-θ-open in X and $x \in f^{-1}(V)$. Let $U = f^{-1}(V)$. Then $f(U) \subseteq V$.
(2) $\rightarrow$ (3): Let $x \in X$ and let V be any regular open subset of Y with $f(x) \in V$. By (2), there exists a b-θ-open set U containing x such that $f(U) \subseteq V$. From Lemma 2.13 there exists a b-open set W such that $x \in W \subseteq bCl(W) \subseteq U$. Since W is b-open, $bCl(W) = bCl_\theta(W)$, and then we have $f(bCl_\theta(W)) \subseteq V$.
(3) $\rightarrow$ (1): This follows from Lemma 2.12(5). □

3. SOME PROPERTIES

Theorem 3.1. *Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$ be the graph function of f. Then, the following properties hold:*

(1) *If g is almost strongly θ-b-continuous, then f is almost strongly θ-b-continuous and X is almost b-regular;*
(2) *If f is almost strongly θ-b-continuous and X is b^*-regular, then g is almost strongly θ-b-continuous.*

Proof. (1) Suppose that g is almost strongly θ-b-continuous. First we show that f is almost strongly θ-b-continuous. Let $x \in X$ and V be a regular open set of Y containing $f(x)$. Then $X \times V$ is a regular open set of $X \times Y$ containing $g(x)$. Since g is almost strongly θ-b-continuous there exists $U \in BO(X, x)$ such that $g(bCl(U)) \subseteq X \times V$. Therefore, we obtain $f(bCl(U)) \subseteq V$. Next we show that X is almost b-regular. Let U be any regular open set of X and $x \in U$. Since $g(x) \in U \times Y$ and $U \times Y$ is regular open in $X \times Y$, there exists $G \in BO(X, x)$ such that $g(bCl(G)) \subseteq U \times Y$. Therefore, we obtain $x \in G \subseteq bCl(G) \subseteq U$ and hence X is almost b-regular.
(2) Let $x \in X$ and W be any regular open set of $X \times Y$ containing $g(x)$. there exist regular open sets $U_1 \subseteq X$ and $V \subseteq Y$ such that $g(x) = (x, f(x)) \in U_1 \times V \subseteq W$. Since f is almost strongly θ-b-continuous, there exists $U_2 \in BO(X, x)$ such that $f(bCl(U_2)) \subseteq V$. Since X is b^*-regular and $U_1 \cap U_2 \in BO(X, x)$, there exists $U \in BO(X, x)$ such that $x \in U \subseteq bCl(U) \subseteq U_1 \cap U_2$ (by Lemma 2.10). Therefore, we obtain $g(bCl(U)) \subseteq U_1 \times f(bCl(U_2)) \subseteq U_1 \times V \subseteq W$. This shows that g is almost strongly θ-b-continuous. □

Lemma 3.2. [13] *If X_0 is α-open in X, then $BO(X_0) = BO(X) \cap X_0$.*

Lemma 3.3. [16] *If $A \subseteq X_0 \subseteq X$, and X_0 is α-open in X, then $bCl(A) \cap X_0 = bCl_{X_0}(A)$, where $bCl_{X_0}(A)$ denotes the b-closure of A in the subspace X_0.*

Theorem 3.4. *If $f : X \rightarrow Y$ is almost strongly θ-b-continuous and X_0 is a α-open subset of X, then the restriction $f|X_0 : X_0 \rightarrow Y$ is almost strongly θ-b-continuous.*

Proof. For any $x \in X_0$ and any regular open set V of Y containing $f(x)$, there exists $U \in BO(X, x)$ such that $f(bCl(U)) \subseteq V$ since f is almost strongly θ-b-continuous. Put $U_0 = U \cap X_0$, then by Lemmas 3.2 and 3.3, $U_0 \in BO(X_0, x)$ and $bCl_{X_0}(U_0) \subseteq bCl(U_0)$.

Therefore, we obtain $(f|X_0)(bCl_{X_0}(U_0)) = f(bCl_{X_0}(U_0)) \subseteq f(bCl(U_0)) \subseteq f(bCl(U)) \subseteq V$. This shows that $f|X_0$ is almost strongly θ-b-continuous. □

Definition 3.5. A space X is said to be b-T_2 (resp; b-Urysohn) [6] if for each pair of distinct points x and y in X, there exist $U \in BO(X, x)$ and $V \in BO(X, x)$ such that $U \cap V = \phi$ (resp; $bCl(U) \cap bCl(V) = \phi$).

Definition 3.6. A space X is said to be rT_0 [1] if for each pair of distinct points x and y in X, there exist regular open set containing one of the points but not the other.

Theorem 3.7. *Let $f : X \to Y$ be injective and almost strongly θ-b-continuous.*

(1) *If Y is rT_0 , then X is b-T_2;*
(2) *If Y is Hausdorff, then X is b-Urysohn.*

Proof. (1) Let x and y be any distinct points of X. Since f is injective, $f(x) \neq f(y)$ and there exists a regular open set V containing $f(x)$ not containing $f(y)$ or a regular open set W containing $f(y)$ not containing $f(x)$. If the first case holds, then there exists $U \in BO(X, x)$ such that $f(bCl(U)) \subseteq V$. Therefore, we obtain $f(y) \notin f(bCl(U))$ and hence $X - bCl(U) \in BO(X, y)$. If the second case holds, then we obtain a similar result. Therefore, X is b-T_2.
(2) As in (1), if x and y are distinct points of X, then $f(x) \neq f(y)$. Since Y is Hausdorff, there exists open sets V and W containing $f(x)$ and $f(y)$ respectively, such that $V \cap W = \phi$. Hence $Int(Cl(V)) \cap Int(Cl(W)) = \phi$. Since f is almost strongly θ-b-continuous, there exist $G \in BO(X, x)$ and $H \in BO(X, y)$ such that $f(bCl(G)) \subseteq Int(Cl(V))$ and $f(bCl(H)) \subseteq Int(Cl(W))$. It follows that $bCl(G) \cap bCl(H) = \phi$. This shows that X is b-Urysohn. □

Lemma 3.8. *Let A be a subset of X and B be a subset of Y. Then*

(1) [13] *If $A \in BO(X)$and $B \in BO(Y)$, then $A \times B \in BO(X \times Y)$.*
(2) [16] $bCl(A \times B) \subset bCl(A) \times bCl(B)$.

Theorem 3.9. *Let $f : X_1 \to Y$, $g : X_2 \to Y$ be two almost strongly θ-b-continuous and Y is Hausdorff, then $A = \{(x_1, x_2) : f(x_1) = g(x_2)\}$ is b-θ-closed in $X_1 \times X_2$.*

Proof. Let $(x_1, x_2) \notin A$. Then $f(x_1) \neq g(x_2)$. Since Y is Hausdorff, there exist open sets V_1 and V_2 containing $f(x_1)$ and $g(x_2)$ respectively, such that $V_1 \cap V_2 = \phi$, hence $Int(Cl(V_1)) \cap Int(Cl(V_2)) = \phi$. Since f and g are almost strongly θ-b-continuous, there exists $U_1 \in BO(X, x_1)$ and $U_2 \in BO(X, x_2)$ such that $f(bCl(U_1)) \subseteq Int(Cl(V_1))$ and $g(bCl(U_2)) \subseteq Int(Cl(V_2))$. Since $(x_1, x_2) \in U_1 \times U_2 \in BO(X_1 \times X_2)$ and $bCl(U_1 \times U_2) \cap A \subseteq (bCl(U_1) \times bC(U_2)) \cap A = \phi$, we have that $(x_1, x_2) \notin bCl_\theta(A)$. Thus A is b-θ-closed in $X_1 \times X_2$. □

In [13], Nasef introduced the notion of B^*-space. If for each $x \in X$, $BO(X, x)$ is closed under finite intersection, then the space X is called B^*-space.

Theorem 3.10. *Let f, g be two almost strongly θ-b-continuous from a B^*-space X into a Hausdorff, space Y. Then the set $A = \{x \in X : f(x) = g(x)\}$ is b-θ-closed.*

Proof. We will show that $X \backslash A$ is b-θ-open. Let $x \notin A$, then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist open sets V_1 and V_2 in Y such that $f(x) \in V_1$ and $g(x) \in V_2$ and $V_1 \cap V_2 = \phi$, hence $Int(Cl(V_1)) \cap Int(Cl(V_2)) = \phi$. Since f and g are almost strongly θ-b-continuous, there exist b-open sets U_1 and U_2 containing x such that $f(bCl(U_1)) \subseteq Int(Cl(V_1))$ and $g(bCl(U_2)) \subseteq Int(Cl(V_2))$.

Take $U = U_1 \cap U_2$. Clearly $U \in BO(X, x)$ because X is B^*-space and $x \in U \subseteq bCl(U) \subseteq bCl(U_1 \cap U_2) \subseteq bCl(U_1) \cap bCl(U_2) \subseteq X \backslash A$ because $f(bCl(U_1)) \cap g(bCl(U_2)) = \phi$. Thus $X \backslash A$ is b-θ-open. □

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\}$ of $X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 3.11. The graph $G(f)$ of a function $f : X \to Y$ is said to be b-θ-closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in BO(X, x)$ and an open set V in Y containing y such that $(bCl(U) \times Cl(V)) \cap G(f) = \phi$.

Lemma 3.12. *The graph $G(f)$ of a function $f : X \to Y$ is b-θ-closed if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in BO(X, x)$ and an open set V in Y containing y such that $f(bCl(U)) \cap Cl(V) = \phi$.*

Theorem 3.13. *Let $f : X \to Y$ be almost strongly θ-b-continuous and Y is Hausdorff, then $G(f)$ is b-θ-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $f(x) \neq y$ Since Y is Hausdorff, there exists open sets V and W in Y containing $f(x)$ and y respectively, such that $Int(Cl(V)) \cap Cl(W) = \phi$. Since f is almost strongly θ-b-continuous, there exist $U \in BO(X, x)$ such that $f(bCl(G)) \subseteq Int(Cl(V))$. Therefore, $f(bCl(U)) \cap Cl(W) = \phi$. and then by Lemma 3.12 $G(f)$ is b-θ-closed in $X \times Y$. □

Recall that a subspace A of X is called a retract of if there is a continuous map $r : X \to A$ (called a retraction) such that for all $x \in X$ and all $a \in A$, $r(x) \in A$, and $r(a) = a$.

Theorem 3.14. *Let A be a subset of X and $r : X \to A$ be almost strongly θ-b-continuous retraction. If X is Hausdorff, then A is b-θ-closed subset of X.*

Proof. Suppose that A is not b-θ-closed. Then there exists a point x in X such that $x \in bCl_\theta(A)$ but $x \notin A$. It follows that $r(x) \neq x$ because r is retraction. Since X is Hausdorff, there exist open sets U and V containing x and $r(x)$ respectively, such that $U \cap V = \phi$, hence $sCl(U) \cap Int(Cl(V)) \subseteq Cl(U) \cap Int(Cl(V)) = \phi$. By hypothesis, there exists $U_* \in BO(X, x)$ such that $r(bCl(U_*)) \subseteq Int(Cl(V))$. Since $U \cap U_* \in BO(X, x)$ and $x \in bCl_\theta(A)$, we have we have $bCl(U \cap U_*) \cap A \neq \phi$. Therefore, there exists a point $y \in bCl(U \cap U_*) \cap A$. So $y \in A$ and $r(y) = y \in bCl(U)$. Since $bCl(U) = sCl(U)$, $sCl(U) \cap Int(Cl(V)) = \phi$ gives $r(y) \notin Int(Cl(V))$. On the other hand, $y \in bCl(U_*)$ and this implies $r(bCl(U_*)) \nsubseteq Int(Cl(V))$. This is contradiction with the hypothesis that r is almost strongly θ-b-continuous retraction. Thus A is b-θ-closed subset of X. □

Theorem 3.15. *Let X ,X_1 and X_2 be topological spaces, If $h : X \to X_1 \times X_2$, $h(x) = (x_1, x_2)$ is almost strongly θ-b-continuous then $f_i : X \to X_i$, $f_i(x) = x_i$ is almost strongly θ-b-continuous for $i = 1, 2$.*

Proof. We show only that $f_1 : X \to X_1$ is almost strongly θ-b-continuous. Let V_1 be any regular open set in X_1. Then $V_1 \times X_2$ is regular open in $X_1 \times X_2$ and hence $h^{-1}(V_1 \times X_2)$ is b-θ-open in X. Since $f_1^{-1}(V_1) = h^{-1}(V_1 \times X_2)$, f_1 is almost strongly θ-b-continuous. □

A subset K of a space X is said to be b-closed relative to X [16] (resp; N-closed relative to X [15]) if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of K by b-open (regular open) sets of X, there exists a finite subset Λ_0 of Λ such that $K \subseteq \cup\{bCl(V_\alpha) : \alpha \in \Lambda_0\}$ (resp; $K \subseteq \cup\{V_\alpha : \alpha \in \Lambda_0\}$).

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Theorem 3.16. *If a function $f : X \to Y$ is almost strongly θ-b-continuous and K is b-closed relative to X,then $f(K)$ is N-closed relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $f(K)$ by regular open sets of Y. For each point $x \in K$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is almost strongly θ-b-continuous there exists $U_x \in BO(X, x)$ such that $f(bCl(U_x)) \subseteq V_{\alpha(x)}$. The family $\{U_x : x \in K\}$ is a cover of K by b-open sets of X and hence there exists a finite subset K_0 of K such that $K \subseteq \cup_{x \in K_0} bCl(U_x)$. Therefore, we obtain $f(K) \subseteq \cup_{x \in K_0} V_{\alpha(x)}$. This shows that $f(K)$ is N-closed relative to Y. □

A topological space X is said to be quasi-H-closed [17] if every cover of X by open sets has a finite subcover whose closures cover X.

Theorem 3.17. *Let X be a submaximal extermally disconnected space. If a function $f : X \to Y$ has a b-θ-closed graph, then $f^{-1}(K)$ is θ-closed in X for each subset K which is quasi-H-closed relative to Y.*

Proof. Let K be a quasi-H-closed set of Y and $x \notin f^{-1}(K)$. Then for each $y \in K$ we have $(x, y) \notin G(f)$ and by Lemma 3.12 there exists $U_y \in BO(X, x)$ and an open set V_y of Y containing y such that $f(bCl(U_y)) \cap Cl(V_y) = \phi$. The family $\{V_y : y \in K$ is an open cover of K and there exists a finite subset K_0 of K such that $K \subseteq \cup_{y \in K_0} Cl(V_y)$. Since X is submaximal extermally disconnected, each U_y is open in X and $bCl(U_y) = Cl(U)$. Set $U = \cap_{y \in K_0} U_y$, then U is an open set containing x and $f(Cl(U)) \cap Cl(K) \subseteq \cup_{y \in K_0}[f(Cl(U)) \cap Cl(V_y)] \subseteq \cup_{x \in K_0}[f(bCl(U_y)) \cap Cl(V_y)] = \phi$. Therefore, we have $Cl(U) \cap f^{-1}(K) = \phi$ and hence $x \notin Cl_\theta(f^{-1}(K))$. This shows that $f^{-1}(K)$ is θ-closed in X. □

Theorem 3.18. *If a function $f : X \to Y$ has a b-θ-closed graph, then $f(K)$ is θ-closed in Y for each subset K which is b-closed relative to X.*

Proof. Let K be a b-closed relative to X and $y \in Y \setminus f(K)$. Then for each $x \in K$ we have $(x, y) \notin G(f)$ and by Lemma 3.12, there exist $U_x \in BO(X, x)$ and open set V_x of Y containing y such that $f(bCl(U_x)) \cap Cl(V_x) = \phi$. The family $\{U_x : x \in K\}$ is a cover of K by b-open sets of X. Since K is b-closed relative to X, there exists a finite subset K_0 of K such that $K \subseteq \cup\{bCl(U_x : x \in K_0)\}$. put $V = \cap\{V_x : x \in K_0\}$. then V is an open set containing y and $f(K) \cap \cap Cl(V) \subseteq [\cup_{x \in K_0} f(bCl(U_x))] \cap Cl(V) \subseteq \cup_{x \in K_0}[f(bCl(U_x)) \cap Cl(V_x)] = \phi$. Therefore, we have $y \in Cl_\theta(f(K))$ and hence $f(K)$ is θ-closed in Y. □

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